1. (15pts) Let $f(z)$ be an entire function with $\mathrm{Im}(f(z)) \geq 2$ for all $z \in \mathbb{C}$. Prove that $f$ is constant.
**Suggestion:** Consider the function $g(z) = e^{if(z)}$.

Write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. From the assumption we have $v(x,y) \geq 2$ for all $x, y \in \mathbb{R}$. Let $g(z) = e^{if(z)}$. By the chain rule $g(z)$ is an entire function. We have

$$|g(z)| = |e^{if(z)}| = |e^{i(u+iv)}| = |e^{iu} e^{-v}| = \overbrace{|e^{iu}|}^{=1} e^{-v}$$
$$= e^{-v} \leq e^{-2}$$

Since $g(z)$ is entire and bounded, $g(z)$ is a constant function by Liouville's theorem. Since $g$ is constant so is $f$.

2. (15 pts) Let $C$ denote the circle $|z| = 3$ oriented counterclockwise. Compute the integral

$$\int_C (\sin(z^2) + \bar{z})\, dz.$$

$$\int_C \sin(z^2) + \bar{z} \; dz = \int_C \sin(z^2)\, dz + \int_C \bar{z}\, dz$$

$\int_C \sin(z^2)\, dz = 0$ since $\sin(z^2)$ is entire and $C$ is a closed curve.

To compute the second integral we use the parametrization

$z = 3e^{i\varphi}$ $\quad 0 \leq \varphi \leq 2\pi$

$\bar{z} = 3e^{-i\varphi}$ and $dz = 3i e^{i\varphi}\, d\varphi$

$$\Rightarrow \int_C \bar{z}\, dz = \int_{\varphi=0}^{2\pi} 3e^{-i\varphi}\, 3i e^{i\varphi}\, d\varphi = 9i \int_{\varphi=0}^{2\pi} d\varphi = 9i(2\pi) = 18\pi i.$$

$$\Rightarrow \int_C \sin(z^2) + \bar{z} \; dz = 18\pi i.$$

3. (15 pts) Consider the polynomial $p(z) = iz^7 + 5z^5 + 1$. Use Rouché's theorem to prove that all zeros of $p$ are contained in the annulus $\frac{1}{2} < |z| < 3$.

Since $\deg(p(z)) = 7$, $p$ has exactly 7 zeros counted with multiplicity by the fundamental theorem of algebra.

On the circle $|z| = 1/2$

we choose $f(z) = 1$ and $g(z) = iz^7 + 5z^5$

$|g(z)| = |iz^7 + 5z^5| \leqslant |z|^7 + 5|z|^5 = \frac{1}{2^7} + \frac{5}{2^5} < \frac{1}{4} + \frac{8}{2^5} = \frac{1}{2} < 1 = |f(z)|$

By Rouché's theorem $f$ and $f+g = p$ have the same number of zeros inside $|z| \leq 1/2$. Since $f$ is constant, it has no zeros, so $p(z)$ has no zeros in $|z| \leqslant 1/2$

On $|z| = 3$, this time we choose $f(z) = iz^7$, and $g(z) = 5z^5 + 1$

$|g(z)| \leq 5|z|^5 + 1 = 5(3^5) + 1 < 6(3^5) < 9(3^5) = 3^7 = |f(z)|$

$f(z)$ and $f(z) + g(z) = p(z)$ have the same number of zeros inside $|z| < 3$, by Rouché's theorem. Since $f$ has 7 zeros, all located at the origin, all 7 zeros of $p$ are also inside $|z| < 3$.

Combining the two resuts we see that all zeros of $p$ are in the annulus $\frac{1}{2} < |z| < 3$.

4. (a) (10 pts) Determine the radius of convergence of the power series

$$h(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^n + 1}$$

(b) (10 pts) Let $\gamma$ denote the unit circle $|z| = 1$ oriented counter clockwise. Find

$$\int_\gamma \frac{h(z)}{z^3}.$$

(a) Write $a_n = \frac{1}{2^n+1}$ so that $h(z) = \sum_{n=0}^{\infty} a_n z^n$

Apply the ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1/(2^{n+1}+1)}{1/(2^n+1)} = \lim_{n\to\infty} \frac{2^n+1}{2^{n+1}+1} = \lim_{n\to\infty} \frac{\cancel{2^n}\,(1+1/2^n)}{\cancel{2^n}\,(2+1/2^n)}$$

$$= 1/2$$

This implies $\limsup |a_n|^{1/n} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1/2.$

and the radius of convergence is $1/\,1/2 = 2.$

(b) $h(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$

$$\frac{h(z)}{z^3} = \frac{a_0}{z^3} + \frac{a_1}{z^2} + \frac{a_2}{z} + a_3 + a_4 z + \cdots$$

$$\Rightarrow \int_C \frac{h(z)}{z^3}dz = 2\pi i \operatorname{Res}\left(\frac{h(z)}{z^3}, 0\right) = 2\pi i\,(a_2) = \frac{2\pi i}{5}$$

5. (a) (10 pts) Let $C_R$ denote the half circle $\{z : |z| = R, \operatorname{Im}(z) \geq R\}$. Prove that

$$\lim_{R\to\infty} \int_{C_R} \frac{z^2}{z^4+1}\, dz = 0$$

(b) (15 pts) Using complex residues, compute the (real valued) integral

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx.$$

(a) for $z \in C_R$ we have $|z| = R$

$\Rightarrow |z^4+1| \geq \left||z|^4 - 1\right| = R^4 - 1$ assuming $R > 1$.

$\Rightarrow \frac{1}{|z^4+1|} \leq \frac{1}{R^4-1}$ and $\left|\frac{z^2}{z^4+1}\right| \leq \frac{R^2}{R^4-1}$

$\Rightarrow \left|\int_{C_R} \frac{z^2}{z^4+1}\, dz\right| \leq \frac{R^2}{R^4-1} \pi R \longrightarrow 0$ as $R \to \infty$.

(b) First notice that the integral $\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx$ converges absolutely since $\frac{x^2}{x^4+1} \geq 0$ $\forall x \in \mathbb{R}$, $\frac{x^2}{x^2+1} \leq \frac{1}{x^2}$ $\forall x \in \mathbb{R}$ and $\int_1^{\infty} \frac{1}{x^2}\, dx$ and $\int_{-\infty}^{-1} \frac{dx}{x^2}$ are both convergent.

Let $f(z) = \frac{z^2}{z^4+1}$ $\forall z \in \mathbb{C}$. Define $L_R : z = x$ for $-R \leq x \leq R$ and let $C_R$ be as in part (a). Let $\Gamma_R = C_R \cup L_R$. We have

$$\int_{\Gamma_R} f(z)\, dz = \int_{C_R} f(z)\, dz + \int_{L_R} f(z)\, dz. \quad (1)$$

$\lim_{R\to\infty} \int_{L_R} f(z) dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^4+1}\, dx = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx$ (2) (what we want to compute)

$\lim_{R\to\infty} \int_{C_R} f(z) dz = 0$ (3) by part (a)

$f(z) = \frac{z^2}{z^4+1}$ has exactly 4 poles $z_1 = e^{\pi i/4}$, $z_2 = e^{3\pi i/4}$, $z_3 = e^{5\pi i/4}$ and $z_4 = e^{7\pi i/4}$. Notice all of these are simple poles.

Among $z_1, z_2, z_3, z_4$ only $z_1$ and $z_2$ lie inside $\Gamma_R$ (for $R>1$)

Hence by the Residue theorem

$$\int_{\Gamma_R} f(z)dz = 2\pi i \left( \operatorname{Res}\left(\frac{z^2}{z^4+1}, z_1\right) + \operatorname{Res}\left(\frac{z^2}{z^4+1}, z_2\right)\right)$$

$$= 2\pi i \left( \frac{z_1^2}{4z_1^3} + \frac{z_2^2}{4z_2^3} \right)$$

$$= \frac{2\pi i}{4} \left( z_1^{-1} + z_2^{-1} \right) = \frac{\pi i}{2}\left(e^{-\pi i/4} + e^{-3\pi i/4}\right)$$

$$= \frac{\pi i}{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}.$$

Since $\int_{\Gamma_R} f(z)dz = \frac{\pi}{\sqrt{2}}$ for all $R>1$, $\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = \frac{\pi}{\sqrt{2}}$. (4)

Combining equations (1), (2), (3) and (4) we get

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}dx = \frac{\pi}{\sqrt{2}}.$$